

The Cox, Clifford, Möbius, Miquel, and other related configurations and their generalizations

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Abstract

The realizability of countable Cox configurations on Miquelian planes is proved. A simple way to determine the automorphisms of the Cox configurations is presented.

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Introduction

The smallest reasonable Cox configuration was invented many years ago, and it appears under various names, depending on the geometry in which it was discovered: Möbius configuration of points and planes, Steiner-Miquel configuration of points, lines, and circles, a suitable biplane; it is also a completion of the Miquel configuration of points and circles/chains. In each case a suitable configuration plays its role in geometry. And it is truly elegant. In the first Section of the paper we briefly explain these matters. Let us mention that even nowadays this configuration is a source of interesting researches which have applications in physics.

Let us consider this smallest Cox configuration as an abstract, purely combinatorial object. It can be, clearly, generalized in various ways depending on the way in which the source configuration is presented. In this note we follow one of these ways, where Cox configurations, defined in Section 2 in terms of elementary combinatorics of finite sets, are associated with the hypercube graphs. In what follows we prove the realizability of Cox configurations on spheres (cf. Thm. 3.7; more generally: in Miquelian chain spaces, in particular: on quadrics); we also discuss relationships of our configurations with generalized Desargues configurations.

Finally, our approach enables us to give an easy way to determine the automorphism group of a Cox configuration (cf. Cor. 4.6).

1 Basic: the Steiner-Miquel configuration

Let us start with a verbal presentation of the classical Steiner-Miquel configuration (cf. [5])¹: Consider a Veblen configuration on the Euclidean plane i.e. four lines

¹this is *our* name, this configuration appears in the literature under quite different names

such that each two distinct of them have a common point. Then four triangles can be formed from these lines (taken as the sides of the triangles in question), and four circles circumscribed on these triangles appear. The circles thus obtained all pass through one point (the so called *Miquel point*). It is one of the classical theorems of so called *Advanced Euclidean Geometry*.

The resulting configuration is much better seen when we retold the whole story on the so called inversive plane: the one-point completion of the Euclidean plane i.e. on a sphere. Then the lines of the considered Veblen configuration are four circles through a fixed point, and the whole configuration appears fully symmetric. Let us present the Steiner-Miquel configuration in this way.

We have eight circles $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ and the points $q_A, q_B, q_{\{i,j\}}$ with $\{i, j\} \in \mathcal{P}_2(\{1, 2, 3, 4\})$ that satisfy the incidences given in Table 1. Clearly, the matrix defines a $(8_4 8_4)$ -configuration $\mathfrak{H} = \langle S, \mathcal{C} \rangle$ in which

- (I) through any pair of points there pass either 0 or 2 blocks,
- (II) no three points are in two blocks,
- (III) each circle has at least three points,
- (IV) a triple of points pairwise on a block is on a block, and
- (V) a triple of pairwise intersecting blocks have a common point.

(cf. axioms for biplanes and semibiplanes, e.g. in [8]). The Steiner-Miquel configuration is visualized in Figure 1. It is self dual (cf. 4.3).

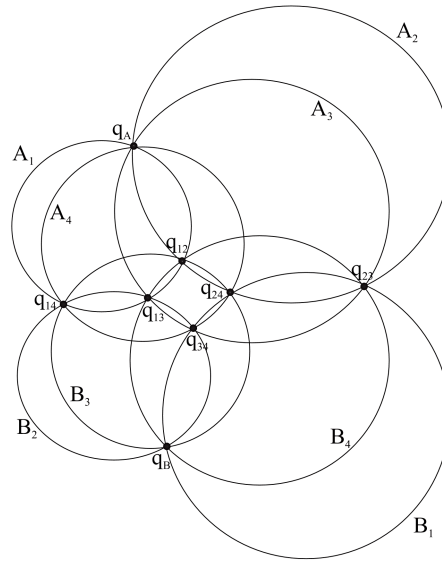
I	q_A	$q_{\{1,2\}}$	$q_{\{1,3\}}$	$q_{\{1,4\}}$	$q_{\{2,3\}}$	$q_{\{2,4\}}$	$q_{\{3,4\}}$	q_B
A_1	×	×	×	×	—	—	—	—
A_2	×	×	—	—	×	×	—	—
A_3	×	—	×	—	×	—	×	—
A_4	×	—	—	×	—	×	×	—
B_1	—	—	—	—	×	×	×	×
B_2	—	—	×	×	—	—	×	×
B_3	—	×	—	×	—	×	—	×
B_4	—	×	×	—	×	—	—	×

Table 1: The incidence matrix of the Steiner-Miquel configuration

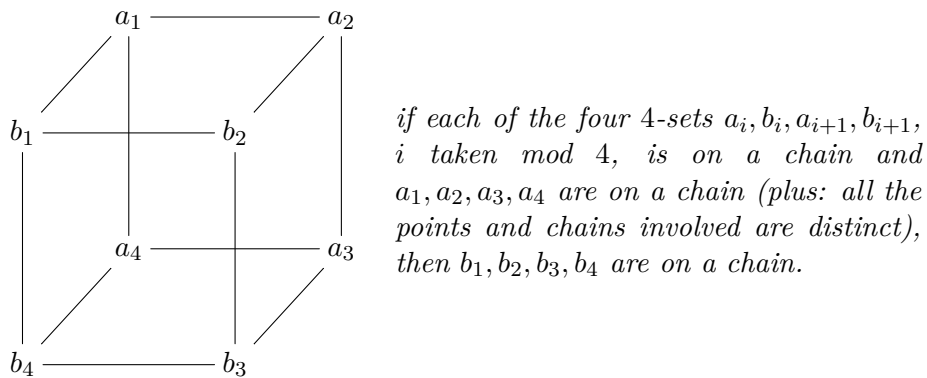
FACT 1.1. *Let $p \in S$ be a point of \mathfrak{H} . Consider the substructure $\mathfrak{H}_{(p)}$ of \mathfrak{H} , whose blocks are the blocks that pass through p and whose points are the points of rank at least two on thus defined blocks with p excluded. Then $\mathfrak{H}_{(p)}$ is the Veblen configuration.*

FACT 1.2. *For each block D of \mathfrak{H} there is a unique block D^\dagger such that D and D^\dagger have no point in common.*

FACT 1.3. *Let $D \in \mathcal{C}$ be arbitrary. The substructure $\langle S, \mathcal{C} \setminus \{D, D^\dagger\} \rangle$ of \mathfrak{H} is the Miquel configuration.*



The Miquel Axiom which states, loosely speaking, that each Miquel configuration closes is considered as a one of fundamental axioms in chain geometry. It is frequently visualized by the schema in Figure 2 ². The “thesis” of this axiom can be rephrased also in the following form, which is more suitable in “weak” chain structures: *if b_1, b_2, b_3 are on a chain B , then $b_4 \in B$.*



An incidence structure $\langle S, \mathcal{C} \rangle$ is called a *weak chain structure* when it satisfies (II), (III), and (IV). Then the following is evident

²For some modern generalizations to higher dimensions see e.g. [7]

Recall also that so called Miquelian Möbius spaces i.e. incidence structures of points and conics on a (projective) sphere (i.e. on a nonruled quadric) in a Pappian projective space are the classical examples of chain structures which satisfy the Miquel axiom.

In view of 1.3, the configuration \mathfrak{H} contains the dual of Miquel Configuration. The so called *novel generalization of Clifford's classical point-circle configuration* (cf. [13, Def. 3.1], [6] is exactly a realization of the dual of the Miquel Configuration on the Euclidean sphere (i.e. in the real Möbius plane). At a first look it is intriguing that ‘novel generalization...’ can be defined as the family of circles circumscribed on the faces of an icosahedron. It becomes less intriguing when we remind that an icosahedron is considered as a dual of a cube, and (see Figure 2) the Miquel Configuration consists of the circles circumscribed on the faces of a cube.

A substructure $\langle S_0, C_0 \rangle$ of an incidence structure $\langle S, C \rangle$ is called a (l_1, l_2) -closed substructure if it meets the following conditions:

- $C \in C, |C \cap S_0| \geq l_1$ imply $C \in C_0$, and
- $|\{C \in C_0 : a \in C\}| \geq l_2$ implies $a \in S_0$.

A $(3, 1)$ -closed substructure of a weak chain structure \mathfrak{M} is frequently referred to as a *subspace* of \mathfrak{M} .

Finally, the configuration \mathfrak{H} is also (the combinatorial scheme of) the Möbius 8_4 -configuration (cf. [4]) consisting of two tetrahedrons simultaneously inscribed and described one in/on the other.

2 The Cox Configuration: definitions

Let X be a set, $|X| \geq 3$. We write $\wp_k(X)$ for the set of k -element subsets of X , for any integer k . The Cox configuration (comp. [4]) is the incidence structure

$$(\mathcal{C}_X)_X := \langle \bigcup \{\wp_{2k}(X) : 0 \leq k \leq |X|\}, \bigcup \{\wp_{2k+1}(X) : 0 \leq k \leq |X|\}, \prec \cup \succ \rangle, \quad (1)$$

where \prec is the direct-successor-relation and $\succ = \prec^{-1}$. We write $(\mathcal{C}_X)_n = (\mathcal{C}_X)_X$ when $n = |X|$. Clearly, if n is finite then the Cox configuration $(\mathcal{C}_X)_n$ is a $(2^{n-1} {}_n 2^{n-1})$ -configuration, and it is a weak chain structure that meets condition (I) for arbitrary at least 3-element set X .

We see that $(\mathcal{C}_X)_4 \cong \mathfrak{H}$ (have a look at Figure 3). The structure $(\mathcal{C}_X)_3$ is visualized in Figure 4. The structure $(\mathcal{C}_X)_5$, much more complicated, being a suitable completion of the Desargues configurations, is presented in Figure 5.

EXAMPLE 2.1. Let X be a set and k be an integer with $1 < k + 1 < |X|$. On the set $\wp_k(X)$ we define two families of its subsets. The sets $T(b) = \{z \in \wp_k(X) : z \subset b\}$ with $b \in \wp_{k+1}(X)$ are called *tops*, and the sets $S(h) = \{z \in \wp_k(X) : h \subset z\}$ with $h \in \wp_{k-1}(X)$ are the *stars*. Let $\mathcal{T}_k(X)$ be the set of all the tops and $\mathcal{S}_k(X)$ be the set of all the stars defined on $\wp_k(X)$. The incidence structure

$$\mathbf{G}_k(X) := \langle \wp_k(X), \mathcal{T}_k(X) \rangle, \quad (2)$$

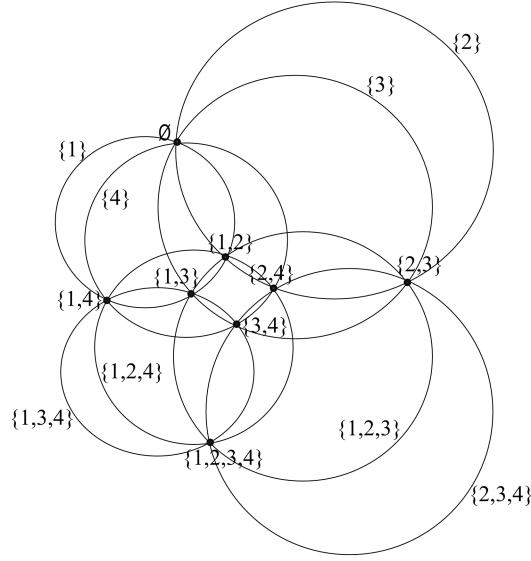


Figure 3: The (classical) Steiner-Miquel configuration labelled by the subsets of $\{1, 2, 3, 4\}$

called a *combinatorial Grassmannian*, is a partial linear space, and for finite $|X| =: n$ it is a $\binom{n}{k}_{n-k} \binom{n}{k+1}_{k+1}$ -configuration (see e.g. [9], [10]). $\mathbf{G}_2(4)$ is the Veblen configuration, $\mathbf{G}_2(5)$ is the Desargues configuration, and from among all the combinatorial Grassmannians the so called *generalized Desargues configurations* $\mathbf{G}_2(X)$ appear most important.

Write

$$\mathbf{K}^\dagger(X, k) := \langle \mathcal{P}_k(X), \mathcal{T}_k(X) \cup \mathcal{S}_k(X) \rangle; \quad (3)$$

The incidence structure $\mathbf{K}^\dagger(X, k)$ is a weak chain structure which satisfies (I). It is

- the structure of the maximal cliques of $\mathbf{G}_k(X)$, and
- a (closed) substructure of $(\mathcal{C}_X)_X$: if k is even, then each block of $(\mathcal{C}_X)_X$ that has two points common with the point set of $\mathbf{K}^\dagger(X, k)$ is actually a block of $\mathbf{K}^\dagger(X, k)$, and dually, when k is odd.

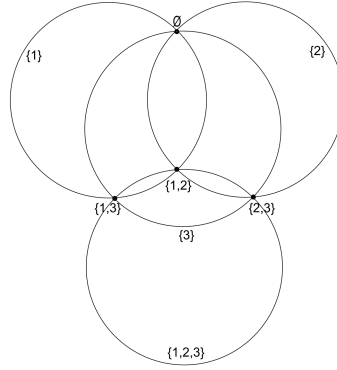
If $n = 2k$ then $\mathbf{K}^\dagger(X, k)$ is a $\binom{2k}{k}_{2k} 2 \binom{2k}{k+1}_{k+1}$ -configuration. \bigcirc

The following is also evident:

FACT 2.2. *If $X_1 \subset X_2$ then $(\mathcal{C}_X)_{X_1}$ is a $((2, 2)$ -closed) subconfiguration of $(\mathcal{C}_X)_{X_2}$.*

FACT 2.3. *The Levi graph of the Cox configuration $(\mathcal{C}_X)_X$ is a hypercube graph.*

Since the hypercube graphs (see [2], [4], [8], and many others) appear one of most “classical” graphs, the above observation explains why Cox configurations deserve some interest also in the context of graph theory. Let us mention, as an example, one observation: *If $8 \leq m \leq n$ and $4|m$, $8|n$ or $8|m$, $4|n$ then the complete bipartite graph $K_{m,n}$ can be decomposed into Q_4 -cubes i.e. $K_{m,n}$ can be presented as the*

Figure 4: $(\mathcal{C}_X)_3$: a “spherical” triangle

union of the Levi graphs of the Möbius M_8 configuration (a consequence of [2, Thm. 2.2]). Other connections of this sort are indicated at the end of Section 3.

The converse of 2.3 is slightly more complex. Let X be arbitrary and \mathfrak{K} be the hypercube graph $\langle \wp(X), \prec \cup \succ \rangle$. If X is infinite then \mathfrak{K} is disconnected. Each two its connected components are isomorphic, and the connected component of \emptyset is the set $\wp_{<\omega}(X)$ of the finite subsets of X . And, in turn, the restriction of \mathfrak{K} to $\wp_{<\omega}(X)$ is the Levi graph of $(\mathcal{C}_X)_X$.

3 Representations and interpretations

The (geometrical) procedure which leads to \mathfrak{H} starting from the Veblen configuration can be easily generalized as below. Just observe the following facts:

FACT 3.1. *Let $u \in \wp_k(X)$ be a point of $(\mathcal{C}_X)_X$. The family of blocks of $(\mathcal{C}_X)_X$ which pass through u with the point u deleted is the generalized Veblen configuration (it consists of $|X| - k$ blocks each two of them having a common point).*

FACT 3.2. *Let k be an integer. Note that the elements of $\wp_{k+1}(X)$ are simultaneously the lines of $\mathbf{G}_k(X)$ and the points of $\mathbf{G}_{k+1}(X)$. A (maximal) clique \mathcal{K} contained in $\wp_{k+1}(X)$ can be either*

- *the set $S(u)$ with $u \in \wp_k(X)$: then \mathcal{K} consists of the bundle of lines of $\mathbf{G}_k(X)$ through a point, or*
- *the set $T(U)$ with $U \in \wp_{k+2}(X)$, a line of $\mathbf{G}_{k+1}(X)$: then \mathcal{K} can be viewed as the lines of the generalized Veblen configuration $\mathbf{G}_k(U)$, and geometrically this family can be visualized as a plane contained in $\mathbf{G}_k(X)$.*

Fact 3.2 enables us to present $(\mathcal{C}_X)_X$ as a suitable completion of the combinatorial Grassmannian $\mathbf{G}_k(X)$.

Representation 3.3. Observe that, formally, $\mathbf{G}_0(X)$ is the structure consisting of the bundle of (one-element) lines through a fixed point, and $\mathbf{G}_{|X|-1}(X)$ is the chain of points on a single line.

Let $2 \mid k$, $0 < k < |X|$. The points of $(\mathcal{C}_X)_X$ in $\wp_k(X)$ are exactly the points of $\mathbf{G}_k(X)$. The lines of $\mathbf{G}_k(X)$ are also blocks of $(\mathcal{C}_X)_X$. With each “plane” U

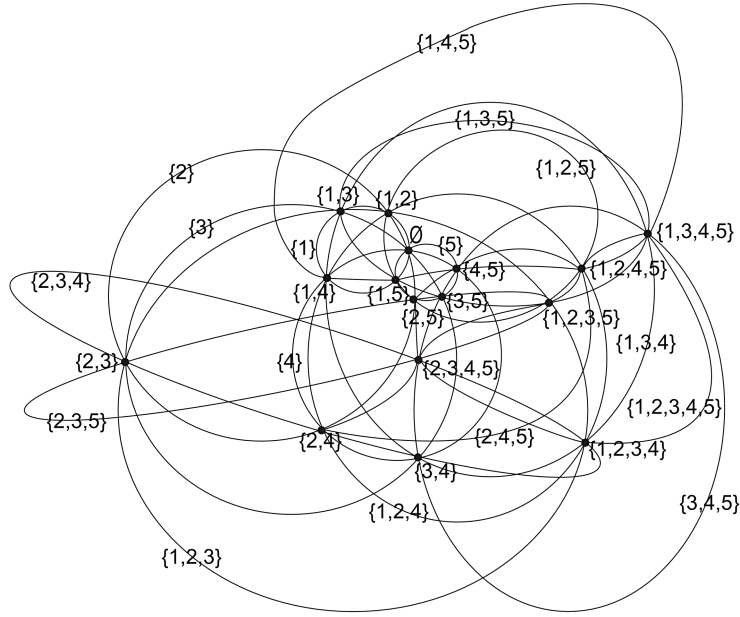


Figure 5: $(\mathcal{C}\chi)_5$: the points labelled by the elements of $\wp_2(\{1, 2, 3, 4, 5\})$ form a Desargues configuration (one can easily check it using the combinatorial representation; it is fairly not obvious observing the figure)

of $\mathbf{G}_k(X)$ we associate a new “improper” point U^∞ and require that all the lines in this plane pass through U^∞ . These improper points correspond to the points of $(\mathcal{C}\chi)_X$ in $\wp_{k+2}(X)$. Clearly, added points are on one block when the planes which determine them have a common line in $\mathbf{G}_k(X)$. With any (maximal) family of planes that do not have a line in common but pairwise are neighbor (if such a family exists in $\mathbf{G}_k(X)$) we add a new block which joins their improper points. The structure of the improper points and new blocks is (isomorphic to) $\mathbf{G}_{k+2}(X)$.

Next, with each maximal clique \mathcal{K} (a simplex) in $\mathbf{G}_k(X)$ we associate a “clique-block” \mathcal{K}° that contains all the points in \mathcal{K} . These clique-blocks correspond to the blocks of $(\mathcal{C}\chi)_X$ in $\wp_{k-1}(X)$. As above, with each family \mathcal{F} of maximal cliques which do not have a common point but pairwise intersect (if it exists in $\mathbf{G}_k(X)$) we associate a new ideal point common to all the clique-blocks \mathcal{K}° with $\mathcal{K} \in \mathcal{F}$. And, as above, the structure of these ideal points and clique-blocks is $\mathbf{G}_{k-2}(X)$.

Continuing in this fashion, finally, we end up with $(\mathcal{C}\chi)_X$. \circ

Presentation 3.3 may be especially useful when applied to a generalized Desargues configuration $\mathbf{G}_2(X)$.

Let us stress on that the original Cox configuration (possibly, it would be better to call it the Steiner-Miquel configuration, as proposed in Section 1), is closely related to the configuration of the Miquel Axiom (see Figure 2), which is nowadays called the Miquel Configuration. And thus it is usually considered as a configuration of circles rather than of planes. It is mentioned in [4] that the configuration $(\mathcal{C}\chi)_X$ can be realized on a 2-sphere for arbitrary set X . Justification of this fact seems incomplete, though. Let us make some comments on the subject.

PROPOSITION 3.4. *The Cox configuration $(\mathcal{C}\chi)_X$ satisfies the Miquel Axiom for every set X .*

PROOF. Let points a_i, b_i satisfy the assumptions of the Miquel Axiom (cf. Figure 2). Without loss of generality (cf. 4.3) we can assume that $a_2 = \emptyset$; then $A = \{k\}$, $a_1 = \{i, k\}$, $a_3 = \{j, k\}$, $a_4 = \{k, l\}$ for some $i, j, k, l \in X$. From the definition of $(\mathcal{C}\chi)_X$ we compute that $b_2 = \{i, j\}$, $b_1 = \{i, l\}$, $b_3 = \{j, l\}$, and $b_4 = \{i, j, k, l\}$, which are on the block $B = \{i, j, l\}$. \square

PROPOSITION 3.5. *Let $(\mathcal{C}\chi)_X$ be realized in a weak chain structure \mathfrak{M} and let $\langle \mathcal{S}, \mathcal{C}_0 \rangle$ be a subspace of \mathfrak{M} which contains a pair of blocks of $(\mathcal{C}\chi)_X$ meeting in two distinct points. Then $(\mathcal{C}\chi)_X$ is contained in \mathcal{S} .*

PROOF. Let us refer to the blocks of \mathfrak{M} as to *circles*. Without loss of generality we can assume that \mathcal{S} contains the circles $\{i_1\}$ and $\{i_2\}$ meeting in \emptyset and $\{i_1, i_2\}$, for some distinct $i_1, i_2 \in X$. Each one of the circles $\{i\}$, $X \ni i \neq i_1, i_2$ crosses our two circles in three distinct points: \emptyset , $\{i_1, i\}$, and $\{i_2, i\}$, and thus $\{i\}$ lies on \mathcal{S} . Consequently, \mathcal{S} contains all the points in $\wp_2(X)$.

Each circle in $\wp_{k+1}(X)$ with $k \geq 2$ contains at least three points in $\wp_k(X)$ so, if $\wp_k(X) \subset \mathcal{S}$, then each circle in $\wp_{k+1}(X)$ lies on \mathcal{S} . Each point in $\wp_{k+1}(X)$ lies on a circle in $\wp_k(X)$, so if the circles in $\wp_k(X)$ all are subsets of \mathcal{S} , we have $\wp_{k+1}(X) \subset \mathcal{S}$. This proves, inductively, that each point and each circle in $\wp(X)$ lies on \mathcal{S} . \square

In an arbitrary Möbius space \mathfrak{M} a pair of chains intersecting in two points determines a plane of \mathfrak{M} which contains them, and the same holds, generally, in chain spaces associated with quadrics (Minkowski spaces, Laguerre, etc.). Thus from 3.5 we directly derive that the configuration $(\mathcal{C}\chi)_X$ is *planar*:

COROLLARY 3.6. *Assume that the configuration $(\mathcal{C}\chi)_X$ is realized in a chain space \mathfrak{M} associated with a quadric (as an important particular case: let \mathfrak{M} be a Möbius space). Then actually it is realized on a plane in \mathfrak{M} .*

So, without loss of generality we can search for a realization of $(\mathcal{C}\chi)_X$ on a 2-sphere.

THEOREM 3.7 (Möbius Representation). *Let X be a finite (at least 3-element) or countable set. The structure $(\mathcal{C}\chi)_X$ can be realized on a 2-sphere \mathcal{S} in a Euclidean 3-space, with the blocks interpreted as circles on \mathcal{S} .*

PROOF. Set $n = |X|$. In view of natural connections of $(\mathcal{C}\chi)_4$ with the Miquel configuration, the claim is evident for $n = 3, 4$. We prove that each realization of $(\mathcal{C}\chi)_{\{1, \dots, n-1\}}$ on a 2-sphere in a Euclidean 3-space can be extended to a realization of $(\mathcal{C}\chi)_{\{1, \dots, n\}}$ (comp. 2.2).

Set $Z = \{1, \dots, n-1\}$, $X = Z \cup \{n\}$ and assume that the configuration $(\mathcal{C}\chi)_Z$ is already realized on a sphere \mathcal{S} , as required. In what follows we shall refer to the subsets of Z as to *points and circles (resp.) of \mathcal{S}* , thus identifying a point/block of the configuration $(\mathcal{C}\chi)_Z$ and its image under a realization: a “real”, “geometrical” point/circle. We are going to construct a realization of $(\mathcal{C}\chi)_X$ on \mathcal{S} extending the given realization of $(\mathcal{C}\chi)_Z$.

Let us choose any circle through \emptyset with the following properties: it lies on \mathcal{S} , is distinct from a , and is not tangent to a for each circle $a \in \wp_1(Z)$; the circle found will be denoted by $\{n\}$. The point of intersection of $\{n\}$ and $\{i\}$ distinct from \emptyset will be denoted by $\{i, n\}$ for each $i \in Z$. This way all the points of $\wp_2(X)$ are already realized on \mathcal{S} .

One can say: proceeding this way we obtain the realization of each point and block of $(\mathcal{C}\chi)_X$. This is a bit hand-waving argument: let us make it more precise. Recall that each point a and each block a of $(\mathcal{C}\chi)_X$ are already realized whenever $n \notin a$.

(i) *Assume that each point a and each block a of $(\mathcal{C}\chi)_X$ with $|a| \leq k$ are properly realized on \mathcal{S} (i.e. they are points and circles on \mathcal{S} that satisfy the incidences required in the definition of $(\mathcal{C}\chi)_X$). Let $n \in p \in \wp_{k+1}(X)$ be a point of $(\mathcal{C}\chi)_X$. Then all the circles $p \setminus \{i\}$ with $i \in p$ have a common point on \mathcal{S} .*

PROOF OF (i). The claim is evident for $k = 2$, so we assume that $k \geq 3$. Let us write $a = p \setminus \{n\}$. Take any distinct $i, j \in a$: we shall prove that a , $p \setminus \{i\}$, and $p \setminus \{j\}$ have a common point (distinct from $a \setminus \{i, j\}$). From assumptions, there is $l \in a$ with $l \neq i, j$. Set $x = a \setminus \{i, j, l\}$ (then x is a point of $(\mathcal{C}\chi)_X$). Observe the schema presented in Figure 6. Between the circles of the diagram there are

$$\begin{array}{ccccc}
 x \cup \{i\} & & - & x, x \cup \{i, j\} & - & x \cup \{j\} \\
 | & & & & & | \\
 x \cup \{i, l\}, x \cup \{i, n\} & & & & & x \cup \{j, l\}, x \cup \{j, n\} \\
 | & & & & & | \\
 x \cup \{i, l, n\} = p \setminus \{j\} & - & x \cup \{l, n\}, q & - & x \cup \{j, l, n\} = p \setminus \{i\}
 \end{array}$$

Figure 6: Illustration to the proof of (i) in 3.7

placed their intersection points. From the inductive assumption of this paragraph, the points x , $x \cup \{i, n\}$, $x \cup \{j, n\}$, and $x \cup \{l, n\}$ are on the circle $x \cup \{n\}$. From the Miquel Axiom we infer that q lies on the circle which contains $x \cup \{i, j\}$, $x \cup \{i, l\}$, and $x \cup \{j, l\}$; the latter circle is a , so a passes through q , as required. \square

(ii) *Assume that each point a and each block a of $(\mathcal{C}\chi)_X$ with $|a| \leq k$ are properly realized on \mathcal{S} . Let $n \in p \in \wp_{k+1}(X)$ be a block of $(\mathcal{C}\chi)_X$. Then all the points $p \setminus \{i\}$ with $i \in p$ lie on a circle on \mathcal{S} .*

PROOF OF (ii). As above, without loss of generality it suffices to assume that $k \geq 4$. Let us write $a = p \setminus \{n\}$. Let $i, j, m \in a$ be pairwise distinct; from assumption, there is $l \in a$ with $l \neq i, j, m$. We shall prove that the points a , $p \setminus \{i\}$, $p \setminus \{j\}$, and $p \setminus \{m\}$ are on a circle on \mathcal{S} . Let $x = a \setminus \{i, j, m, l\}$ and consider the diagram in Figure 7: From inductive assumption, the points $x \cup \{m, l\}$, $x \cup \{j, l\}$, $x \cup \{i, l\}$, and $x \cup \{l, n\}$ are on the circle $x \cup \{l\}$. From the Miquel Axiom we get the desired claim. \square

In view of the above we can inductively extend given realization from the elements of $\wp_k(X)$ to the elements of $\wp_{k+1}(X)$ thus obtaining a required embedding. \square

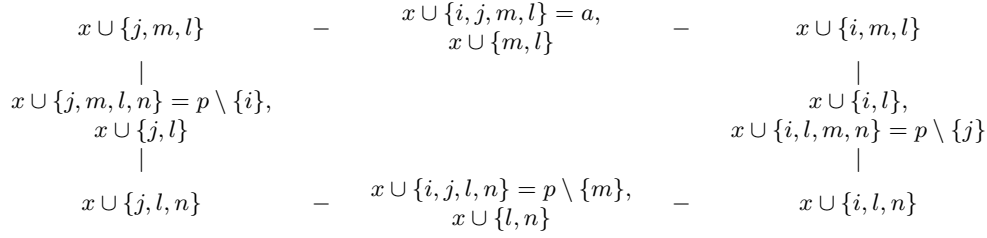


Figure 7: Illustration to the proof of (ii) in 3.7

This, what is perhaps more interesting is an observation, which follows from the way in which our spherical representation was defined:

COROLLARY 3.8. *Let \mathfrak{M} be a Miquelian Benz plane (i.e. let it be a sphere, a hyperboloid, or a cylinder in a Pappian projective 3-space, with the family of conics on it distinguished) such that the size of its chain is at least $n + 1$. The structure $(\mathcal{C}\chi)_n$ can be realized on \mathfrak{M} , with the blocks interpreted as chains of \mathfrak{M} .*

COROLLARY 3.9. *For each positive integer n the structure $(\mathcal{C}\chi)_n$ can be realized on a hyperboloid and on a cylinder in the real projective 3-space.*

Theorem 3.7 for infinite countable X was already remarked in [1]. Note that the configuration $(\mathcal{C}\chi)_n$ realized on a real 2-sphere (in more modern language: on the real Möbius plane) is referred also as a *Clifford Configuration* (cf. [11], [12], [14] and many others). Babbage gives in [1] for an infinite Clifford Configuration the labelling of its points and circles by the finite sets of positive integers, which directly shows that it is isomorphic to the Cox configuration, no connection like this is indicated, though.

Let X be an arbitrary set and $A \subset X$. Write $\tau_A(Z) = A \div Z = (A \setminus Z) \cup (Z \setminus A)$ for any set Z . Slightly generalizing 2.2 we see that if $X = X_1 \cup X_2$ is finite, $X_1 \cap X_2 = \emptyset$, and $A \in \wp(X_{3-i})$, $i \in \{1, 2\}$ then the set

$$\tau_A((\mathcal{C}\chi)_{X_i}) = \{\tau_A(a) : a \in \wp(X_i)\}$$

yields a subconfiguration $\mathcal{C}\chi^A$ of $(\mathcal{C}\chi)_X$ isomorphic to $(\mathcal{C}\chi)_{X_i}$. Particularly, $\tau_\emptyset((\mathcal{C}\chi)_{X_i})$ is simply $(\mathcal{C}\chi)_{X_i}$ naturally embedded in $(\mathcal{C}\chi)_X$ (comp. 2.2). The two families $(\mathcal{C}\chi)^i = \{\mathcal{C}\chi^A : A \in \wp(X_{3-i})\}$ $i = 1, 2$ are mutually transversal and cover the underlying Cox configuration $(\mathcal{C}\chi)_X$ in such a way that each its flag is a flag of one of the configurations $\mathcal{C}\chi^A$, $A \in \wp(X_1) \cup \wp(X_2)$. On one hand this observation allows us to decompose a Cox configuration into smaller, simpler Cox configurations. E.g. it enables us to imagine $(\mathcal{C}\chi)_8$ as a union of $2^4 + 2^4 = 32$ Möbius configurations $(\mathcal{C}\chi)_4$, $(\mathcal{C}\chi)_6$ as a union of $2^3 + 2^3 = 16$ ‘spherical triangles’ $(\mathcal{C}\chi)_3$, or $(\mathcal{C}\chi)_9$ as a union of 2^5 Möbius configurations and 2^4 configurations $(\mathcal{C}\chi)_5$. On the other hand these decompositions correspond to decompositions of the (suitable) cube graphs (cf. 2.3) into smaller cube graphs (Q_d decomposed into $Q_{d_1}, Q_{d_2}, \dots, Q_{d_t}$, where $d = d_1 + \dots + d_t$).

4 Automorphisms of Cox Configurations

The following are immediate.

LEMMA 4.1. *Let $\varphi \in S_X$ and let $\tilde{\varphi}$ be the natural extension of φ to a bijection of $\wp(X)$. Clearly, the map $\varphi \mapsto \tilde{\varphi}$ is a group monomorphism $S_X \longrightarrow S_{\wp(X)}$.*

Then $\tilde{\varphi} \in \text{Aut}((\mathbb{C}\chi)_X)$ for each $\varphi \in S_X$.

LEMMA 4.2. *Let $A \in \wp(X)$ be arbitrary finite. Then*

$$\tau_A \text{ is } \begin{cases} \text{an automorphism of } (\mathbb{C}\chi)_X & \text{when } 2 \mid |A|, \\ \text{a correlation of } (\mathbb{C}\chi)_X & \text{when } 2 \nmid |A|. \end{cases}$$

As a consequence of 4.2 we get an evident

COROLLARY 4.3. *The structure $(\mathbb{C}\chi)_X$ is homogeneous and self-dual for any set X . Moreover, its automorphism group is flag-transitive.*

Any two stabilizers $(\text{Aut}((\mathbb{C}\chi)_X))_A$ with $A \in \wp(X)$ are isomorphic.

LEMMA 4.4. *The action of $(\text{Aut}((\mathbb{C}\chi)_X))_\emptyset$ on the set $\wp_{<\omega}(X)$ coincides with the action of the group $\{\tilde{\varphi}: \varphi \in S_X\}$ on this set.*

PROOF. Let $f \in (\text{Aut}((\mathbb{C}\chi)_X))_\emptyset$, then f yields a permutation of the blocks through \emptyset ; right from the definition this means that there is $\varphi \in S_X$ such that $f(\{i\}) = \{\varphi(i)\} = \tilde{\varphi}(\{i\})$ for each $i \in X$. Next, f maps the points of intersection of the above blocks onto itself, particularly, $f(\{i, j\}) = \{\varphi(i), \varphi(j)\} = \tilde{\varphi}(\{i, j\})$ for each pair i, j of distinct elements of X . Continuing in this fashion we end up with $f(a) = \tilde{\varphi}(a)$ for each $a \in \wp_k(X)$, for every positive integer k . \square

Let $\varphi \in S_X$, $A \subset X$. Clearly,

$$\tilde{\varphi} \circ \tau_A \circ \tilde{\varphi}^{-1} = \tau_{\tilde{\varphi}(A)},$$

Summing up 4.1, 4.2, and 4.4 we conclude with

COROLLARY 4.5. *The group of collineations and correlations of $(\mathbb{C}\chi)_X$ is isomorphic to the semidirect product $S_X \ltimes \langle \wp_{<\omega}(X), \div \rangle$. In the particular case $|X| = n < \omega$, the group of collineations and correlations of $(\mathbb{C}\chi)_n$ is thus isomorphic to the semidirect product $S_n \ltimes C_2^n$.*

Evidently, the set $\{A \subset X: 2 \mid |A| < \infty\}$ is a subgroup of the 2-group $\langle \wp(X), \div \rangle$. This yields immediately (cf. also [4])

COROLLARY 4.6. *The group of collineations of $(\mathbb{C}\chi)_X$ is isomorphic to the semidirect product $S_X \ltimes \langle \{a \in \wp_{<\omega}(X): 2 \mid |a|\}, \div \rangle$.*

In particular, $\text{Aut}((\mathbb{C}\chi)_n) \cong S_n \ltimes C_2^{n-1}$

Remark 1. Let $|X| = 2k$ and $k > 2$. Then $\text{Aut}(\mathbf{K}^\dagger(X, k)) \cong C_2 \oplus S_X$. This can be directly deduced from known characterizations of the automorphisms of combinatorial Grassmannians, and of the maps which preserve binary collinearity of points in such Grassmannians. In view of 4.1 and 4.2 this means also that each automorphism of $\mathbf{K}^\dagger(X, k)$ can be extended to an automorphism of $(\mathbb{C}\chi)_X$: an automorphism of $\mathbf{K}^\dagger(X, k)$ is (a restriction of) a map of the form $(\tau_X)^\varepsilon \circ \tilde{\varphi}$, where $\varphi \in S_X$, and $\varepsilon = 0, 1$.

A characterization of the automorphisms of geometrical Cox and Clifford configurations is not so evident: an automorphism of a configuration \mathfrak{D} realized in another geometrical structure \mathfrak{M} is commonly required to be a “geometrical” automorphisms i.e. it is required to be an *automorphism of \mathfrak{M}* which leaves \mathfrak{D} invariant. Classification of such automorphisms of \mathfrak{D} may depend on the way in which \mathfrak{D} is embedded into \mathfrak{M} (comp. e.g. [3] in the case of automorphisms of the Desargues Configuration on the projective plane). In any case, a geometrical automorphism of \mathfrak{D} is an (ordinary) automorphism of \mathfrak{D} as well. Let us have a look at the following example.

EXAMPLE 4.7. Let $\mathfrak{D} = (\mathcal{C}\chi)_n$ be realized on a Möbius plane \mathfrak{M} and $\varphi \in S_n$. A geometrical automorphism $\tilde{\varphi}$ (if it exists) permutes circles through a fixed point. In particular, for every circle $\{i\}$ the harmonic ratios $(\emptyset, \{i, j_1\} // \{i, j_2\}, \{i, j_3\})$ and $(\emptyset, \{\varphi(i), \varphi(j_1)\} // \{\varphi(i), \varphi(j_2)\}, \{\varphi(i), \varphi(j_3)\})$ should coincide for each three-element set $\{j_1, j_2, j_3\} \subset \{1, \dots, n\} \setminus \{i\}$. Thus, e.g., the permutation $(1)(2)(3, 4)(5) \in \text{Aut}((\mathcal{C}\chi)_5)$ does not extend to any geometrical automorphism. Slightly more generally: *If $\text{id} \neq \varphi \in S_X$, $|\text{Fix}(\varphi)| \geq 3$ then $\tilde{\varphi}$ is never a geometrical automorphism of \mathfrak{D} .*

A systematic study of the admissible sizes of the groups of geometrical automorphisms of the Cox (Clifford) configurations is addressed to other papers. \circ

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